MATH 147 QUIZ 10 SOLUTIONS

1. Let C be the curve with parameterization $\mathbf{r}(t) = (\cos(t), \sin(t), t)$, $0 \leq t \leq 2\pi$ so that C is that portion of the helix of radius one from $(1,0,0)$ to $(1,0,1)$. Find a second parameterization of C and use this to create a reparameterization of C. Then, check that $\int_C x + y + z \; ds$ is independent of the two parameterizations. (6 Points)

   We see that another parameterization of the curve C is given by $\mathbf{s}(t) = (\cos(2t), \sin(2t), 2t)$ for $0 \leq t \leq \pi$. We then calculate the two integrals. Note that $\mathbf{r}'(t) = (-\sin(t), \cos(t), 1)$, so $\|r'(t)\| = \sqrt{\sin^2(t) + \cos^2(t) + 1} = \sqrt{2}$. Then,

$$\int_C x + y + z \; ds = \sqrt{2} \int_0^{2\pi} \cos(t) + \sin(t) + t = \sqrt{2} \left[\sin(t) - \cos(t) + \frac{t^2}{2} \right]_0^{2\pi} = 2\sqrt{2}\pi^2.$$

   On the other hand, one has $\mathbf{s}(t) = (-2\sin 2t, 2\cos 2t, 2)$, giving $\|s'(t)\| = \sqrt{8} = 2\sqrt{2}$. Using this, one has

$$\int_C x + y + z \; ds = 2\sqrt{2} \int_0^{\pi} \cos(2t) + \sin(2t) + 2t \; dt = 2\sqrt{2} \left[\frac{1}{2}\sin(2t) - \frac{1}{2}\cos(2t) + t^2 \right]_0^{\pi} = 2\sqrt{2}\pi^2.$$

2. For the sphere $S : x^2 + y^2 + z^2 = 4$, find the plane tangent to S at $P = (1, 1, \sqrt{2})$. (4 points)

   To find the tangent plane, we first parameterize our surface. Since this is a sphere, we use spherical coordinates with $\rho = 2$. This gives us the parameterization $S(\theta, \varphi) = (2\cos(\theta)\sin(\varphi), 2\sin(\theta)\sin(\varphi), 2\cos(\varphi))$ As $z = \sqrt{2} = 2\cos(\varphi)$, we have the corresponding point $(\pi/4, \pi/4)$ in (θ, φ) plane. Then, the tangent vectors are

$$T_\theta = (-\sqrt{2}\sin(\theta), \sqrt{2}\cos(\theta), 0) \to T_\theta(\pi/4) = (-1, 1, 0)$$

   and

$$T_\varphi = (\sqrt{2}\cos\varphi, \sqrt{2}\cos\varphi, -2\sin\varphi) \to T_\varphi(\pi/4) = (1, 1, -\sqrt{2})$$

   . This allows us to find a normal vector to the plane with a cross product.

$$\begin{bmatrix} i & j & k \\ -1 & 1 & 0 \\ 1 & 1 & -\sqrt{2} \end{bmatrix} = (-\sqrt{2}, -\sqrt{2}, -2).$$

   Note that we expect this to point up since we were at $(1, 1, \sqrt{2})$, so we flip the sign. Finally, we take the dot product of the normal vector with $(x - 1, y - 1, z - \sqrt{2})$ to get the plane

$$\sqrt{2}(x - 1) + \sqrt{2}(y - 1) + 2(z - \sqrt{2}) = 0$$

   or equivalently

$$x + y + \sqrt{2}z = 4.$$